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RIGIDITY IN THE ELASTOPLASTIC TORSION OF SIMPLE RODS*

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Prismatic rods for which the trajectories of tangential stresses under elastic deformation are close to the known trajectories of these stresses in the limiting case of perfect plasticity, are considered. Attention is given to the study of the rigidity of the elastoplastic torsion in the case of perfect plasticity.

The problem of the pure torsion of prismatic inelastic rods occupies a special place among the boundary value problems of mechanics of continuous media, even though it is the simplest of its class; if we exclude the case in which the yield drop is present /1/, then the torsion will not be accompanied by relief of stress; the limiting case of perfectly plastic torsion is statically determinable and can be studied using elementary methods. The appearance of partial plastic deformation formally complicates the problem /2/. However, it is usually the values of the deformation that are of practical interest and not the stresses. It is the deformations that often set a limit to the admissible loads. It is clear that in this context the torsional rigidity is of overriding interset. It can be determined very accurately in an indirect manner, by passing the solution of the partial differential equation at the unknown elastoplastic boundary. The elastic torsion of thin-walled and cylindrical rods when there are no stress concentration foci is investigated in a fairly simple manner in /3/. Plastic deformation reduces the sharpness of the stress concentration and thus widens the range of applicability of the simplified methods of solving elastoplastic problems more efficiently, the higher the level of plastic deformations as compared with elastic deformations. At the centre of the proposed simplification lies the idea of determining the tangential stress trajectories at the periphery of the transverse cross-section in the region of maximum load for elastic as well as the plastic materials; on the contour itself they are identical by virtue of the boundary conditions (the contour is always a trajectory of tangential stresses). The greatest difference between the trajectories under elastic and plastic deformations will occur in the case of perfect plasticity. Nevertheless, the error in determining the torsional rigidity when the actual tangential stress trajectories in the elastic stage are replaced by the trajectories for a perfectly plastic material is practically nil for all singly connected rods with a convex contour, and when parts of the contour are indented with the radius of curvature of the indentations exceeding the distance to the nearest point of the branch of the contour lying opposite /3/. The magnitude of this error represents "the measure of simplicty" of the rod under torsion, and the upper limit of the error when determining the torsional rigidity of the inealstic rods. The more plastic the material (i.e. the greater the plastic deformations), the smaller the error in determining the torsional rigidity; in the limiting case of infinitely large deformations without reinforcement it tends to zero. The present paper deals with the case of linear reinforcement, but the computations are carried out for perfect plasticity.

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1. Pure torsion corresponds to the case when the transverse cross-sections rotate in their planes as rigid figures about the longitudinal Oz-axis by the angle θ, and there are longitudinal displacements $w(x, y)$. Let us find the components of the deformation tensor in a rectangular system of coordinates (l, n, z) under the conditions given above, at any point (x, y)

$$\gamma_{lz} = \partial w/\partial l + \theta H_l(x, y), \quad \gamma_{nz} = \partial w/\partial n + \theta H_n(x, y) \tag{1.1}$$

Here $H_l(x, y)$ is the distance between the centre of rotation (from the oz-axis) and ray l emerging from the point (x, y), and $H_n(x, y)$ is defined in exactly the same manner.

Let us determine the circulation of the deformation tensor components (γ) along the contour $ABDE$ (Fig.1). We have

$$\int_{ABDE} \gamma_s\, ds = 2\theta\omega_{ABDE}$$

where $ABDE$ is a closed contour belonging to the transverse cross section and ω_{ABDE} is the area bounded by this contour. Taking into account the fact that the above contour is formed by the elements of tangential stress trajectories (AB, DE) and elements of the lines orthogonal to them, we can write the last equation in the form

$$\gamma_{AB}\cdot AB - \gamma_{DE}\cdot DE = 2\theta\omega_{ABDE}$$

Here we express the shear deformations in terms of the stresses beyond the elastic limit (τ_T is the torsional yield point, G is the shear modulus and G' is the corresponding tangential modulus). Here we find that

$$\gamma = \tau_T/G + (\tau - \tau_T)/G' \tag{1.2}$$

Here we find that

$$\tau_{AB}\cdot AB - \tau_{DE}\cdot DE + (AB - DE)(G' - G)\,\tau_T/G = 2G'\theta\omega_{ABDE}$$

Using the notation

$$\tau^\circ = \tau + (G' - G)\tau_T/G = G'\gamma \tag{1.3}$$

we obtain

$$\overset{\circ}{\tau}_{AB}\cdot AB - \overset{\circ}{\tau}_{DE}\cdot DE = 2G'\theta\omega_{ABDE} \tag{1.4}$$

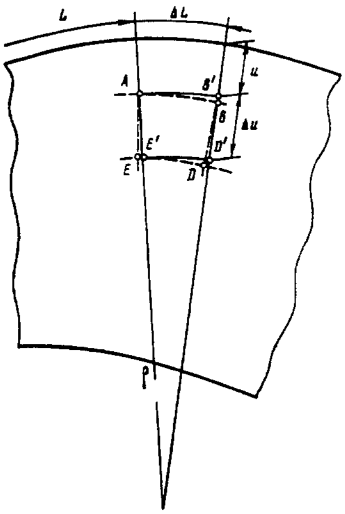

Fig.1

We will place every point of the contour of the transverse cross section of the rod into correspondence with a circle touching the contour at the given point and contained completely within the transverse cross section, with the largest radius equal to $(\delta/2)$. We shall call the segments of the straight lines connecting the centre of this circle with the points on the contour which the circle touches, the conjugated normals. The position of the point B' (Fig.1) can be found if the normal on which the point lies is known, as well as the distance u to the contour line $(0 \leqslant u \leqslant \delta/2)$.

We will define the position of the normal by the length of the contour segment (L) contained between some fixed point and the normal in question. We will assume, to be specific, that the positive direction of the curvilinear coordinate L is the one for which the passage around the contour of transverse cross section is such that the area of the cross section always lies to the right, and the coordinate u is reckoned in the direction of the inner normal to the contour line.

We will assume that the normals in question are sufficiently close to the lines orthogonal to the trajectories of tangential stresses (in the case of perfect plasticity the lines coincide)

$$\tau^\circ(u)\,\frac{\rho - u}{\rho}\,dL - \tau^\circ(u + du)\,\frac{\rho - u - du}{\rho}\,dL \approx 2G'\theta\,\frac{\rho - u - \eta\, du}{\rho}\,dL\,du \tag{1.5}$$

where ρ is the radius of curvature of the element dL (ρ is assumed positive for the convex part of the contour), $0 < \eta < 1$. The last equation yields

$$\frac{\partial\tau^\circ}{\partial u} - \frac{1}{\rho - u}\,\tau^\circ = -2G'\theta$$

The general solution of this equation has the form

$$\tau^\circ = \frac{c}{\rho - u} + G'\theta(\rho - u), \quad c = \mathrm{const}$$

Using the notation (1.3) we obtain

$$\tau = \tau_T - \frac{G'}{G}\tau_T + \frac{c}{\rho - u} + G'\theta(\rho - u) \tag{1.6}$$

where c is the constant of integration obtained from the condition of coupling between the elastic and plastic zone (u° is the coordinate of the elastoplastic boundary)

$$\tau(L, u)|_{u=u^\circ} = \tau_T \tag{1.7}$$

As a result, we write the general solution in the form

$$\tau = \tau_T\left(1 - \frac{G'}{G} + \frac{G'}{G}\frac{\rho - u^\circ}{\rho - u}\right) + G'\theta\frac{(\rho-u)^2 - (\rho - u^\circ)^2}{\rho - u} \tag{1.8}$$

Putting $G' = G$ in (1.6) we obtain the equation characterising the stress field in the elastic kernel

$$\tau = \frac{c}{\rho - u} + G\theta(\rho - u)$$

Expressing the constants in terms of the stress $\tau(L, \delta/2)$ which we shall denote by τ_c, we obtain (here and henceforth the upper sign is taken in the case when the previously chosen directions of the contour stress $\tau(L, 0)$ and the stress τ_c coincide)

$$\tau = \pm\tau_c\frac{\rho - \delta/2}{\rho - u} + G\theta\left[\rho - u - \frac{(\rho - \delta/2)^2}{\rho - u}\right] \tag{1.9}$$

We will call the following expression the Prandtl function (the flux of tangential stresses):

$$\Pi(u) = \int_0^u \tau_n\,du, \quad 0 \leqslant u \leqslant u^\circ \tag{1.10}$$

$$\Pi(u) = \int_0^{u^\circ} \tau_n\,du + \int_{u^\circ}^{u} \tau_y\,du, \quad u^\circ \leqslant u \leqslant \delta/2$$

where τ_n and τ_y denote the stresses in the plastic and elastic zone, given by (1.8) and (1.9), respectively. By requiring that the Prandtl function be continuous at the point with coordinate $u = \delta/2$, we obtain

$$\int_0^{u^\circ} \tau_n\,du + \int_{u^\circ}^{\delta/2} \tau_y\,du = \int_0^{u_*^\circ} \tau_n\,du + \int_{u_*^\circ}^{\delta/2} \tau_y\,du \tag{1.11}$$

where u° and u_*° are the coordinates of the points of intersection of the elastoplastic boundary with the conjugated normals.

Expressions (1.8) and (1.9) together with conditions of continuity (1.11) and (1.7), form a closed system of equations which yields the functions $\tau(L, \delta/2)$ and $u^\circ(L)$, and these in turn define the Prandtl function. The moment of the internal forces is found from the formula (F is the area of transverse cross section)

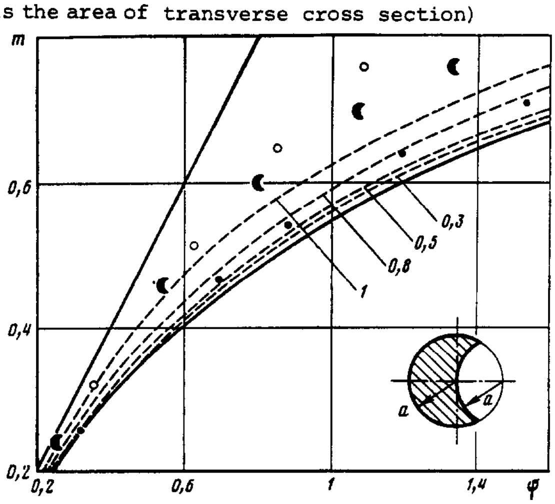

Fig.2

$$M = 2\iint_{(F)} \Pi(L, u)\,dF \tag{1.12}$$

2. In the case of perfect plasticity the maximum tangential stress is equal to the yield point ($\tau_n = \tau_T$) everywhere in the plastic zone, and directed perpendicular to the normals of the contour line. The stress field in the elastic nucleus is found from (1.9). The Prandtl function in this case takes the form

$$\Pi(u) = \tau_T u, \quad 0 \leqslant u \leqslant u^\circ \tag{2.1}$$

$$\Pi(u) = \tau_T u^\circ + \int_{u^\circ}^{u} \tau_y\,du, \quad u^\circ \leqslant u \leqslant \delta/2$$

Expression (1.12), taking (2.1) into account, determines the perfectly elastic plastic torsional moment for the given twist. When the plastic zone has spread over the whole area of the transverse cross section, the torsional moment takes its limiting value

$$M_n = \frac{1}{4}\tau_T \oint_L \left(\delta^2 - \frac{\delta^3}{3\rho}\right) dL \qquad (2.2)$$

The relation connecting the torsional moment M with the twist θ (M_y and θ_y are limiting values of the corresponding quantities, and K is the torsional rigidity within the elastic limits)

$$m = \frac{M - M_y}{M_n - M_y}, \quad \varphi = K\frac{\theta - \theta_y}{M_n - M_y}$$

is shown in Fig.2. The dashed lines show the values of $m(\varphi)$ for the rectangular cross sections, with numbers accompanying the lines describing the ratios of the sides. The half-moons correspond to the Weber profile shown in the same figure. The small circles show the results for the corresponding cross sections with a ratio of the internal to external radii equal to 0.9, and the dark circles refer to the values of m for rods of circular transverse cross section.

The computations show that the curves determining the relationship $m(\varphi)$ are contained within the zone shown in Fig.2 with thick lines. When the twist θ is increased, the plastic zones in which the assumptions made hold exactly, also increase and the magnitude of the torsional moment tends to its exact value (2.2).

We note that the largest error in determining the torsional moment using the formulas given occurs at the yield point. Comparing the values of M_y with the known exact solutions we find that the maximum error is small in the case of simple rods. For prismatic rods of elliptical and rectangular cross section the error does not exceed 3% and 5%, respectively, and for the Weber profile (Fig.2) 1.5%. We note that the torsional moment of a rod of rectangular cross section is determined in /4/ using a more complicated method, yet achieving the same accuracy as in the present paper for a ratio of the sides equal to 0.2 and 0.4. In /5/ the results for a square transverse cross section fall below the limit curve and cannot therefore be regarded as possible.

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EXTENSION OF THE VARIATIONAL FORMULATION OF THE PROBLEM FOR A RIGID-PLASTIC MEDIUM TO VELOCITY FIELDS WITH SLIP-TYPE DISCONTINUITIES*

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Sets of velocity fields containing slip-type discontinuities at the boundary of the rigid-plastic medium, as well as within it, and the functionals defined on these sets, are described. It is shown that the exact lower bounds of the variational problems for these functionals are equal to the coefficient of the critical load. The minimax problem with saddle point constructed here is regarded as an extension of the classical minimax problem of the theory of critical loads.

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